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A Finiteness Theorem of Meromorphic Maps into a Compact Normal Complex Space

Dedicated to Professor Y. Kusunoki for his 60th birthday

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Abstract. Let L be a holomorphic line bundle over an N-dimensinal compact normal complex space X. Take divisors $D_1, \cdots, D_{N+2} \in |L|$ and $E_1, \cdots, E_{N+2}$ on a normal complex space Y. We shall prove that the set of all algebraically nondegenerate meromorphic maps f of Y into X such that $f^*(D_i)=E_i (1 \leqq i \leqq N+2)$ is finite if Y is a compact complex space minus an irreducible analytic subset and $D_1, \ldots, D_{i-1}, D_{i+1}, \ldots, D_{N+2}$ are algebraically independent for each i.

§1. Introduction.

In 1928, H. Cartan proved that, if three non-constant meromorphic functions φ, ψ and χ on C have the same inverse images with multiplicities counted for three distinct values, then $\varphi=\psi$ or $\psi=\chi$ or $\chi=\varphi$([2] , [8]). Relating to this, the author showed in his paper [5] that, for hyperplanes $H_1, \cdots, H_{N+2}$ in $P^N(\mathbf{C})$ located in general position and effective divisors $E_1, \cdots, E_{N+2}$ on $\mathbf{C}^n$, the set of all nondegenerate meromorphic maps f of $\mathbf{C}^n$ into $P^N(\mathbf{C})$ satisfying the condition $f^*(H_i)=E_i (1 \leqq i \leqq N+2)$ is finite, where $f^*(H_i)$ denotes the pull-back of H_i considered as a divisor by f. Afterwards, he gave a generalization of this to the case of meromorphic maps of $\mathbf{C}^n$ into a compact complex manifold ([6]). The purpose of this note is to give a generalization of these results to the case of meromorphic maps of a compact normal complex space minus an irreducilbe analytic subset into another compact normal complex space.

Let X be an N-dimensional compact connected normal complex space and L be a holomorphic line bundle over X. We denote the set of all holomorphic sections of L by $H^0(X, \mathcal{O}(L))$ and the set of all divisors $((\phi))$ associated with zeros of holomorphic sections ϕ of L by $|L|$.

DEFINITION 1.1. Take nonzero holomorphic sections $\phi_1, \cdots, \phi_m$ of L and set $D_i=(\phi_i)(1 \leqq i \leqq m)$. We say that $\phi_1, \cdots, \phi_m$ (or $D_1, \cdots, D_m$) are *algebraically independent with respect to* L if there exists no nonzero homogeneous polynomial $P(w_1, \cdots, w_m)$ such that

$$P(\phi_1, \cdots, \phi_m)\equiv O$$

in $H^0(X, \mathcal{O}(L^d))$, where $d=\deg P$.

DEFINITION 1.2. A meromorphic map f of a normal connected complex space Y into X is called *algebraically nondegenerate with respect to* L if $f(Y) \subsetneqq \{\phi=0\}$ for any nonzero holomorphic section ϕ of L^d, where d is a positive integer.

Let L be a holomorphic line bundle over a compact connected nomal complex space X of dimension N and Y be a compact connected normal complex space minus an irreducible analytic subset. For divisors $D_1, \cdots, D_{N+2} \in |L|$ and effective divisors $E_1, \cdots, E_{N+2}$ on Y, we consider the set $\mathcal{F}(E_i\,;\, D_i)$ of all meromorphic maps f of Y into X which are algebraically nondegenerate with respect to L and satisfy the condition $f^*(D_i)=E_i(1\leqq i\leqq N+2)$.

We now state the main result of this note.

MAIN THEOREM. *If $D_1, \cdots, D_{i-1}, D_{i+1}, \cdots, D_{N+2}$ are algbraically independent with respect to L for each $i(1\leqq i\leqq N+2)$, then $\mathcal{F}(E_i\,;\, D_i)$ is finite.*

More precisely, the number of the elements of $\mathcal{F}(E_i\,;\, D_i)$ is shown to be bounded by a constant depending only on L.

We give some elementary properties on divisors in §2 and prove in §3 a finiteness theorem of meromorphic maps of a complex space minus a thin analytic set into $P^N(\mathbf{C})$ which is nondegenerate in some sense. Main Theorem is completely proved in §4.

§2. Preliminaries on divisors.

In the following sections, by a complex space we mean a σ-compact connected normal complex space. We denote the set of all singularities of X by $S(X)$ and set $R(X):=X-S(X)$. As is well-known, $S(X)$ is an analytic subset of X with codim $S(X)\geqq 2$. We consider the set $\mathcal{V}(X)$ of all 1-codimensional irreducible analytic subsets of X.

DEFINITION 2.1. A *divisor* D on X is a map $D: \mathcal{V}(X)\rightarrow \mathbf{Z}$ which is locally finite, namely, which satisfies the condition that each $x\in X$ has a neighborhood U such that

$$\#\{V\in \mathcal{V}(X)\,;\, U\cap V\neq\phi,\ D(V)\neq 0\}<\infty,$$

where $\mathbf{Z}$ is the set of all integers and $\#S$ denotes the number of the elements of a set S. The *support* of a divisor D is defined by

$$\mathrm{supp}(D)=\cup\ \{V\in \mathcal{V}(X)\,;\, D(V)\neq 0\}$$

For a divisor D on X the set $\mathcal{V}_D:=\{V\in\mathcal{V}(X)\,; D(V)\neq 0\}$ is at most countable. If $\mathcal{V}_D=\{V_i\,; i=1, 2, \cdots\}$, we write $D=\Sigma_i\, n_i V_i$, where $n_i=D(V_i)$. In the case $\mathcal{V}_D=\phi$, we write $D=0$. A divisor $D=\Sigma_i\, n_i V_i$ is called *effective* if $n_i\geqq 0$ for every i. Let U be an open set in X. The restriction of $D=\Sigma_i\, n_i V_i$ to U is defined by

$$D|U:=\Sigma_{i,j}\ n_i\ V_{ij}$$

when each $V_i \cap U$ has the irreducible decomposition $V_i \cap U = \cup_j V_{ij}$.

Let X, Y be complex spaces and f be a many-valued map of X into Y, where $f(x)$ is a subset of Y for each $x \in X$.

DEFINITION 2.2. We say that f is a *meromorphic map* of X into Y if the graph

$$G^f := \{(x,y) \in X \times Y \ ; y \in f(x)\}$$

satisfies the conditions ;

(i) G^f is an irreducible analytic subset of $X \times Y$,

(ii) the projection $\pi_X \mid G^f : G^f \rightarrow X$ is proper,

(iii) $\pi_X \mid \pi_X^{-1}(U) : \pi_X^{-1}(U) \rightarrow U$ is bijective for a non-empty open subset U of X.

For a meromorphic map $f : X \rightarrow Y$ we denote by I_f the set of all points $x \in X$ such that $\# f(x) > 1$. Then, I_f is an analytic set in X with codim $I_f \geqq 2$ and $f \mid X - I_f : X - I_f \rightarrow Y$ may be considered as a single-valued map.

On the N-dimensional complex projective space $P^N(\mathbf{C})$, taking homogeneous coordinates $(w_1 : \cdots : w_{N+1})$ arbitrarily, we set $H_{N+1} = \{w_{N+1} = 0\}$. By identifying a point $(z_1, \cdots, z_N) \in \mathbf{C}^N$ with $(z_1 : \cdots : z_N : 1) \in P^N(\mathbf{C})$ we may consider $\mathbf{C}^N = P^N(\mathbf{C}) - H_{N+1}$ and $\mathbf{C} \cup \{\infty\} = P^1(\mathbf{C})$ in the particular case $N=1$, where $\infty = (1 : 0)$.

PROPOSITION 2.3. (i) *Let f be a meromorphic map of X into $P^1(\mathbf{C})$ with $f \not\equiv \infty$. Then, for each point $x_0 \in X$ there exist holomorphic functions φ, ψ on an open neighborhood U of x_0 such that $\psi \not\equiv 0$ and $f(x) = (\varphi(x) : \psi(x))$ for every $x \in U - \{\psi = 0\}$.*

(ii) *For arbitrary holomorphic funtions φ, $\psi (\not\equiv 0)$ there exists a meromorphic map $f : U \rightarrow P^1(\mathbf{C})$ with $f \not\equiv \infty$ such that $f(x) = (\varphi(x) : \psi(x))$ for every $x \in U - \{\varphi = \psi = 0\}$.*

PROOF. The assertion (i) is obvious on $R(X) - I_f$. It holds on the totality of X because codim $(S(X) \cup I_f) \geqq 2$(cf., [7], p. 243). To see (ii), it suffices to take the map f whose graph is the closure of the set

$$\{(x, (\varphi(x) : \psi(x))) \ ; x \in U, (\varphi(x), \psi(x)) \neq (0,0)\} .$$

By definition, a meromorphic function on X is a meromorphic map $f : X \rightarrow P^1(\mathbf{C})$ with $f \not\equiv \infty$.

In the same manner, we can easily show

PROPOSITION 2.4. *Every meromorphic map $f : X \rightarrow P^N(\mathbf{C})$ with $f(X) \subsetneqq H_{N+1}$ is written as*

$$(2.5) \quad f(x) = (\varphi_1(x) : \cdots : \varphi_N(x) : 1)$$

outside a thin analytic set with meromorphic functions $\varphi_1, \cdots, \varphi_N$ on X. Conversely, for arbitrary meromorphic functions $\varphi_1, \cdots, \varphi_N$ there exists a meromorphic map $f : X \rightarrow P^N(\mathbf{C})$ satisfying (2.5).

Let φ be a nonzero holomorphic function on an open subset U of X. For each $x \in R(X) \cap U$, taking holomorphic local coordinates $z = (z_1, \cdots, z_n)$ with $x = (0)$, we expand φ as

$$\varphi(z) = \sum_{m=0}^{\infty} P_m(z)$$

around x, where each $P_m(z)$ is a homogeneous polynomial of degree m or vanishes identically. We then define $\nu_\varphi(x)=\min\ \{m\ ;\ P_m(z)\not\equiv 0\}$, which does not depend on the choice of holomorphic local coordinates. We set $Z := \{x\in U\ ;\ \varphi(x)=0\}$ and consider the irreducible decomposition $Z=\cup_i Z_i$. As is easily seen, $\nu_\varphi(x)$ is equal to a constant m_i on $R(Z_i)\cap R(X)$ for each i. We define the divisor D_φ associated with φ by $D_\varphi := \Sigma_i m_i Z_i$.

DEFINITION 2.6. A divisor D on X is called a *C-divisor* if, for each point $x\in X$, on a neighborhood U of x D can be written as $D\,|\,U=D_\varphi$ with a nonzero holomorphic function φ on U.

Let $f: X\to Y$ be a meromorphic map and D be a C-divisor on Y such that $f(X)\subsetneqq$ supp (D). For every $x\in X-I_f$ we take a neighborhoods U of x in X and a neighborhood V of $f(x)$ in Y such that $f(U)\subseteqq V$ and $D|\,V=D_\varphi$ for a nonzero holomorphic function φ on V. Then, $\varphi\circ f\ |\ U$ is a nozero holomorphic function and the divisor $D_{\varphi\circ f}$ does not depend on the choice of the above φ. We can define a divisor D_* on $X-I_f$ such that $D_*\,|\,U=D_{\varphi\circ f}$ for each holomorphic function φ on U with the above properties. Let $D_*=\Sigma_i n_i V_i$ on $X-I_f$. Since codim $I_f\geqq 2$, $\bar{V}_i\in\mathscr{V}(X)$ and the family $\{\bar{V}_i\}$ is locally finite.

DEFINITION 2.7. We call the divisor $f^*(D)=\Sigma_i n_i\bar{V}_i$ the *pull-back* of D by f.

Let f be a nonzero meromophic function on X. It is easily seen that there exists exactly one divisor D_f on X such that $D_f\,|\,U=D_\varphi-D_\psi$ for nonzero holomorphic functions φ, ψ on an open set U satisfying the identity $f=(\varphi:\psi)$ outside a thin analytic set.

PROPOSITION 2.8. *For two nonzero meromorphic functions f_1 and f_2, $D_{f_1}=D_{f_2}$ if and only if there exists a nowhere vanishing holomophic function h such that $f_2=hf_1$.*

PROOF. Since codim $S(X)\geqq 2$, $D_{f_1}=D_{f_2}$ on X if and only if $D_{f_1}=D_{f_2}$ on $R(X)$. On the other hand, for a meromorphic function $h := f_2/f_1$, $D_{f_1}|\,R(X)=D_{f_2}\,|\,R(X)$ if and only if h is holomorphic and has no zero on $R(X)$. This implies that $\{h=0\}=\{h=\infty\}=\phi$, because otherwise they are of pure codimension one and so are not included in $S(X)$.

§3. A finiteness theorem of meromorphic maps into $P^N(\mathbf{C})$.

Let $\tilde{X}$ be a complex space, A be a thin analytic subset and $X := \tilde{X}-A$. We denote the field of all meromorphic functions on X by $M(X)$ and the set of all nowhere vanishing holomorphic functions by $H^*(X)$. We denote also the set of all meromorphic functions on X which have meromorphic extensions to $\tilde{X}$ by $M(\tilde{X})$.

DEFINITION 3.1. Take an arbitrary field K such that $M(\tilde{X})\cap H^*(X)\subseteqq K\subseteqq M(\tilde{X})$. We say that a meromorphic map $f: X\to P^N(\mathbf{C})$ is *K-nondegenerate* if $f(X)\subsetneqq P^N(\mathbf{C})\cap\{w_{N+1}=0\}$ and, for meromorphic functions $\varphi_1,\cdots,\varphi_N$, 1 are linearly independent over K.

Take a hyperplane

$$H: a_1w_1+\cdots+a_{N+1}w_{N+1}=0$$

in $P^N(\mathbf{C})$. Then, we can define a C-divisor (H) on $P^N(\mathbf{C})$ such that $(H)\ |\ U_i=D_{\alpha_i}$ for each $i(1\leqq i\leqq N+1)$, where $U_i=P^N(\mathbf{C})\cap\{w_i\neq 0\}$ and $\alpha_i=\Sigma_{j=1}^{N+1} a_j(w_j/w_i)$.

PROPOSITION 3.2. *Let f, $g: \tilde{X}\to P^N(C)$ be K-nondegenerate meromorphic maps for a field K with $M(\tilde{X})\cap H^*(X)\subseteqq K\subseteqq M(\tilde{X})$. If there exist hyperplanes $H_1,\cdots,H_{N+2}$ in $P^N(C)$*

located in general position such that $f^*(H_i)=g^*(H_i)+D_{\alpha_i}$ *for some* $\alpha_i\in K(1\leqq i\leqq N+2)$, *then* $f=g$.

PROOF. We can choose homogeneous coordinates $(w_1:\cdots:w_{N+1})$ on $P^N(\mathbf{C})$ such that $H_i=\{w_i=0\}$ $(1\leqq i\leqq N+1)$ and $H_{N+2}=\{w_1+\cdots+w_{N+1}=0\}$. By Proposition 2.4, we can find meromorphic functions φ_i and ψ_i on $\tilde{X}$ such that $f=(\varphi_1:\cdots:\varphi_{N+1})$ and $g=(\psi_1:\cdots:\psi_{N+1})$, where $\varphi_{N+1}=\psi_{N+1}=1$. Set $\varphi_{N+2}=\varphi_1+\cdots+\varphi_{N+1}$ and $\psi_{N+2}=\psi_1+\cdots+\psi_{N+1}$. Then, by assumption

$$D_{\varphi_i}=f^*(H_i)-f^*(H_{N+1})=g^*(H_i)-g^*(H_{N+1})+D_{\alpha_i/\alpha_{N+1}}$$

$$=D_{\psi_i}+D_{\alpha_i/\alpha_{N+1}}$$

on X for $i=1,2,\cdots,N+2$. Set $h_i=\psi_i\alpha_i/\varphi_i\alpha_{N+1}$, which is by Proposition 2.8 a nowhere vanishing holomorphic function on X and has a meromorphic extension to $\tilde{X}$. Therefore, we see $\tilde{h}_i := h_i(\alpha_{N+1}/\alpha_i)\in K$. We then have a linear relation

$$(\tilde{h}_{N+2}-\tilde{h}_1)\varphi_1+\cdots+(\tilde{h}_{N+2}-\tilde{h}_{N+1})\varphi_{N+1}=0.$$

By the assumption of K-nondegeneracy of f, this implies that $\tilde{h}_{N+2}=\tilde{h}_1=\cdots=\tilde{h}_{N+1}$. We have thus $f=g$.

Now, we give a finiteness theorem of meromorphic maps into $P^N(\mathbf{C})$.

THEOREM 3.3. *Let* $H_1,\cdots,H_{N+2}$ *be hyperplanes in* $P^N(C)$ *located in general position,* $E_1,\cdots,E_{N+2}$ *be effective divisors on* X *and* $\mathcal{F}$ *be the set of all* K*-nondegenerate meromorphic maps of* X *into* $P^N(C)$ *such that* $f^*(H_i)=E_i+D_{\alpha_i}$ *for some* $\alpha_i\in K(1\leqq i\leqq N+2)$, *where* K *is a field with* $M(\tilde{X})\cap H^*(X)\subseteqq K\subseteqq M(\tilde{X})$. *Then,* $\#\mathcal{F}$ *is bounded by a constant depending only on* N.

For the proof, we recall some results in the previous papers [4] ~[6].

THEOREM 3.4. *Let* $f_1,\cdots,f_p\in H^*(X)$ $(p\geqq 2)$ *such that* $f_i/f_j\notin M(\tilde{X})$ *for every* $i,j(i\neq j)$. *Then,* $f_1,\cdots,f_p$ *are linearly independent over* $M(\tilde{X})$.

PROOF. We prove this by induction on p. This is obvious for the case $p=2$. Under the assumption that this is true for the case $\leqq p-1$ we shall prove Theorem 3.4. Suppose that $\alpha_1f_1+\cdots+\alpha_pf_p=0$ for $\alpha_1,\cdots,\alpha_p\in M(\tilde{X})$. Our purpose is to show $\alpha_1=\cdots=\alpha_p=0$. Changing indices if necessary, we may assume that f_1/f_2 has essential singularities along an irreducible component A_0 of A. Then, we see codim $A_0=1$. Take an arbitrary point $x_0\in R(A)\cap R(X)$ and choose holomorphic local coordinates $z_1,\cdots,z_n$ around x_0 in $\tilde{X}$ such that $x_0=0$, $U^n:=\{|z_1|<1,\cdots,|z_n|<1\}\subset\tilde{X}$ and $U^n\cap A=U^n\cap\{z_1=0\}$. We divide the set $I=\{1,2,\cdots,p\}$ of indices into disjoint subclasses $I_1,\cdots,I_k$ such that, for $i\in I_l$ and $j\in I_m$, $f_i/f_j\in M(U^n)$ if and only if $l=m$. We may assume $l\in I_l(1\leqq l\leqq K)$, where $k\geqq 2$ by assumption. Then, we have

$$\textstyle\sum^k_{l=1}(\sum_{i\in I_l}\alpha_i(f_i/f_l))f_l=0.$$

This implies that

$$\sum_{i\in I_l}\alpha_i f_i=0 \ (1\leqq l\leqq k)$$

on U^n by the use of Theorem 4.1 of [4], where we take $m_i=\infty$. By the theorem of identity, they hold on the totality of X. Since $\# I_l\leqq p-1$ for each l, we have $\alpha_i=0$ for all $i=1,2,\cdots,p$.

COROLLARY 3.5. *Suppose that $h_1,\cdots,h_t\in H^*(X)$ satisfy the condition $h_1^{l_1}h_2^{l_2}\cdots h_t^{l_t}\notin M(\tilde{X})$ for every $l_1,\cdots,l_t\in \boldsymbol{Z}$ with $(l_1,\cdots,l_t)\neq(0,\cdots,0)$. Then, $P(h_1,\cdots,h_t)\not\equiv 0$ for every nonzero polynomial $P(w_1,\cdots,w_t)$ with coefficients in $M(\tilde{X})$.*

PROOF. We set

$$P(w_1,\cdots,w_t)=\Sigma'\alpha_{l_1\cdots l_t}w_1^{l_1}\cdots w_t^{l_t},$$

where $\alpha_{l_1\cdots l_t}\in M(\tilde{X})$ and Σ' means the sum through indices $l_1,\cdots,l_t$ with $\alpha_{l_1\cdots l_t}\neq 0$. Apply Theorem 3.4 to functions $f_{l_1\cdots l_t}:=h_1^{l_1}\cdots h_t^{l_t}\in H^*(X)$. The identity

$$P(h_1,\cdots,h_t)=\Sigma'\alpha_{l_1\cdots l_t}h_1^{l_1}\cdots h_t^{l_t}=0$$

contradicts the conclusion of Theorem 3.4.

We now consider an (abstract) field K of characteristic zero. By $K[u]$ we denote the ring of all polynomials in u with coefficients in K. Take pq monomials

$$P_{ji}(u)=\alpha_{ji}u^{l_{ji}}\in K[u] \quad (1\leqq i\leqq p,\ 1\leqq j\leqq q),$$

where l_{ji} are nonzero integers and $\alpha_{ji}\neq 0$.

PROPOSITION 3.6. *For each $q_0(\leqq 1)$ there exists some constant $Q(p,q_0)$ depending only on p and q_0 with the following property :*

If the number q of rows of the matrix $(P_{ji}(u))$ is larger than $Q(p,q_0)$ and

$$\text{rank}\,(P_{ji}(u)\ ;\ 1\leqq i\leqq p,\ 1\leqq j\leqq q)<p,$$

then, after suitable changes of the orders of the indices $i=1,2,\cdots,p$ and $j=1,2,\cdots,q$ respectively, there exists an integer r with $2\leqq r\leqq p$ such that

$$l_{1i}-l_{1i'}=l_{2i}-l_{2i'}=\cdots=l_{q_0 i}-l_{q_0 i'}$$

for all i, i' with $1\leqq i<i'\leqq r$ and

$$\text{rank}\,(P_{ji}(u)\ ;\ 1\leqq i\leqq r,\ 1\leqq j\leqq q_0)<r.$$

For the proof, refer to [6]. The argument in [6], §3 are available to the proof of Proposition 3.6 if we replace the field $\mathbf{C}$ by K.

PROOF OF THEOREM 3.3. Taking homogeneous coordinates $(w_1:\cdots:w_{N+1})$ on $P^N(\mathbf{C})$ suitably, we may identify $P^N(\mathbf{C})$ with the subspace

$$H_0:=\{(w_1:\cdots:w_{N+2})\ ;\ w_1+\cdots+w_{N+2}=0\}$$

of $P^{N+1}(\mathbf{C})$ and H_i with $H_0\cap\{w_i=0\}$ $(1\leqq i\leqq N+2)$. For convenience' sake, we set $p=N+2$ in the following.

Assume that $\mathscr{F}$ contains at least q distinct maps, say $f^1, \cdots, f^q$. We express each $f^j (1 \leqq j \leqq q)$ as

$$(3.7) \quad f^j = (f_1^j : \cdots : f_p^j)$$

with meromorphic functions f_i^j on X, where we may take $f_{i_0}^j = 1$ for some i_0. Then, by assumption and Proposition 2.8, we can find $h_{ji} \in K \cdot H^*(X)$ such that $f_i^j = h_{ji} f_i^1$. Since $f^j(X) \subseteqq \mathrm{H}_0$, we have

$$h_{j1} f_1^1 + h_{j2} f_2^1 + \cdots + h_{jp} f_p^1 = 0 \ (1 \leqq j \leqq q).$$

This implies that

$$(3.8) \quad \det (h_{ji} \ ; \ i = 1, 2, \cdots, p, \ j = j_1, \cdots, j_p) = 0$$

for every $j_1, \cdots, j_p$ $(1 \leqq j_l \leqq q)$.

We may regard $H^*(X)$ as a multiplicative group and $H^*(X) \cap M(\tilde{X})$ as its subgroup. The factor group $G := H^*(X)/H^*(X) \cap M(\tilde{X})$ is torsion free because, for $h \in H^*(X)$ and a nonzero integer l, $h^l \in M(\tilde{X})$ implies $h \in M(\tilde{X})$. Using this fact, we can choose $\eta_1, \cdots, \eta_t \in H^*(X)$ such that $\eta_1^{l_1} \cdots \eta_t^{l_t} \in M$ $(\tilde{X})$ for any $(l_1, \cdots, l_t) = (0, \cdots, 0)$ and each h_{ji} is represented as

$$h_{ji} = \alpha_{ji} \eta_1^{l_{ji}^1} \cdots \eta_t^{l_{ji}^t} \ (1 \leqq i \leqq \mathrm{p}, \ 1 \leqq j \leqq q)$$

for some $\alpha_{ji} \in K$ and $l_{ji} \in \mathbf{Z}$. Set $\boldsymbol{l}_{ji} = (l_{ji}^1, \cdots, l_{ji}^t) \in \mathbf{Z}^t$. We can find integers $p_1, \cdots, p_t, q_j$ such that, for $l_{ji} = l_{ji}^1 p_1 + \cdots + l_{ji}^t p_t + q_j$, $l_{ij} \geqq 0$ and $l_{ki} - l_{ki'} = l_{k'i} - l_{k'i'}$ if and only if $\boldsymbol{l}_{ki} - \boldsymbol{l}_{ki'} = \boldsymbol{l}_{k'i} - \boldsymbol{l}_{k'i'}$ $(1 \leqq i, \ i' \leqq p, \ 1 \leqq k, \ k' \leqq q)$.

We set $P_{ji}(u) := \alpha_{ji} u^{l_{ji}} \in K[u]$. Then,

$$\operatorname{rank} (P_{ji}(u) \ ; \ 1 \leqq i \leqq p, \ 1 \leqq j \leqq q) < p.$$

In fact, by (3.8),

$$(3.9) \quad \det (\alpha_{ji} \eta_1^{l_{ji}^1} \cdots \eta_t^{l_{ji}^t} \ ; \ i = 1, \cdots, p, \ j = j_1, \cdots, j_p) = 0$$

for every $j_1, \cdots, j_p (1 \leqq j_l \leqq q)$. This is a polynomial identity with coefficients in K and indeterminates $\eta_1, \cdots, \eta_t$ as a result of Corollary 3.5. Thus we obtain the desired conclusion by substituting $\eta_i = u^{p_i}$ in (3.9).

Now, we apply Proposition 3.6 to these monomials and then we have the following assertion.

(3.10) If $q > Q(p, q_0)$, after a suitable change of indices we can find holomorphic functions $k_1, \cdots, k_r \in H^*(X)$ such that $\gamma_{ji} := h_{ji}/(h_{j1} k_i) \in K$ for $i = 1, \cdots, r$ and $j = 1, 2, \cdots, q_0$, and

$$\det (h_{ji} \ ; \ i = 1, 2, \cdots, r, \ j = j_1, \cdots, j_r) = 0$$

for every $j_1, \cdots, j_r$ with $1 \leqq j_1, \cdots, j_r \leqq q_0$, where $2 \leqq r \leqq p$.

In fact, the conclusion

$$l_{1i}-l_{1i'}=\cdots=l_{q_0 i}-l_{q_0 i'}\ (1\leqq i\leqq i'\leqq r)$$

of Proposition 3.6 implies that

$$\boldsymbol{l}_{1i}-\boldsymbol{l}_{1i'}=\cdots=\boldsymbol{l}_{q_0 i}-\boldsymbol{l}_{q_0 i'}.$$

Set $(\tilde{l}^1_{i'},\cdots,\tilde{l}^t_{i}) := \boldsymbol{l}_{1i}-\boldsymbol{l}_{11}$ and define $k_i := \eta_1^{\tilde{l}_i^1}\cdots\eta_t^{\tilde{l}_i^t}$, which satisfy the desired condition.

Instead of (3.7) we may use representations $f^j=(f_1^j h_{1j}^{-1} : f_2^j h_{1j}^{-1} : \cdots : f_p^j h_{1j}^{-1})$ for $1\leqq i\leqq r$ and $1\leqq j\leqq q_0$. Then, each original h_{ij} is replaced by h_{ij}/h_{1j}. If we set $\tilde{k}_i=k_i f_i^1$, then we have

(3.11) $f^j=(\gamma_{j1}\tilde{k}_1 : \cdots : \gamma_{jr}\tilde{k}_r : f_{r+1}{}^j : \cdots : f_p{}^j)(1\leqq j\leqq q_0)$,

where $\gamma_{ji}\in K$ and rank $(\gamma_{ji}\ ;\ 1\leqq i\leqq r,\ 1\leqq j\leqq q_0) < r$.

By the induction on N we shall show that there exists a number q_N depending only on N such that $\#\mathscr{F}\leqq q_N$. We first study the case $N=1$. Suppose that $\mathscr{F}$ contains $q := Q(3,3)+1$ distinct maps $f^1,\cdots,f^q$. We may assume that f^j can be written as (3.11), where $q_0=3$. Since $\gamma_{12}\neq\gamma_{22}$, there is no possibility of $\mathrm{rank}(\gamma_{ij})\leqq 1$ and so $r=3$. Then, the identities

$$\gamma_{11}\tilde{k}_1+\gamma_{12}\tilde{k}_2+\gamma_{13}\tilde{k}_3=0,$$

$$\gamma_{21}\tilde{k}_1+\gamma_{22}\tilde{k}_2+\gamma_{32}\tilde{k}_3=0$$

imply that $\tilde{k}_1/\tilde{k}_3$ and $\tilde{k}_2/\tilde{k}_3$ are contained in $M(\tilde{X})$. This concludes that f^1 and f^2 have meromorphic extensions to $\tilde{X}$. This contradicts Proposition 3.2. Therefore, we can conclude $\#\mathscr{F}\leqq q_1 := Q(3,3)$.

Now, we assume that Theorem 3.3 is true for the case $\leqq N-1$ and so there exist numbers $q_1,\cdots,q_{N-1}$ with the desired property where we may assume $q_j<q_{j+1}$ $(1\leqq j\leqq N-2)$. Suppose that $\mathscr{F}$ contains q distinct maps $f^1,\cdots,f^q$. Then, we shall show that $q\leqq q_N := Q(N+2,\ q_{N-1}+1)$. We may assume that each f^j has a representation (3.11), where $q_0=q_{N-1}+1$. We first consider the case $r=N+2$. We may assume $s :=$ rank $(\gamma_{ij}\ ;\ 1\leqq i\leqq r,\ 1\leqq j\leqq q_0)=p-1$. For, otherwise, r may be replaced by $s+1$. Then, the identities

$$\gamma_{j1}\tilde{k}_1+\gamma_{j2}\tilde{k}_2+\cdots+\gamma_{jp}\tilde{k}_p=0\ (1\leqq j\leqq N+1)$$

imply that $\tilde{k}_i/\tilde{k}_p\in M(\tilde{X})$ for $i=1,2,\cdots,N+1$. This shows that $f^1,\ \cdots,\ f^{N+1}$ have meromorphic extensions to $\tilde{X}$, which contradicts Proposition 3.2. Therefore, $r\leqq N+1$. Then, after changing indices, we can choose some $\lambda_2,\ \cdots,\ \lambda_r\in K$ such that

$$\gamma_{j1}=\Sigma_{i=2}^r\gamma_{ji}\lambda_j\ (\mathrm{j}=1,2,\cdots,q_0).$$

Set $k_i^* := \tilde{k}_i+\lambda_i\tilde{k}_1$ for $i=2,\cdots,r$, where $k_i^*\neq 0$ by assumption. We now define mutually distinct meromorphic maps

$$\tilde{f}^j=(\gamma_{j2}k_2^* : \cdots : \gamma_{jr}k_r^* : f^j_{r+1} : \cdots : f^j_p)$$

of X into $P^N(\mathrm{C})$. It is easily seen by assumption that each $\tilde{f}^j$ is K-nondegenerate and $\tilde{f}^j(X) \subsetneqq \{w_2+\cdots+w_p=0\}$. Therefore, setting $H_i=\{w_i=0\} \cap P^N(\mathrm{C})$ $(2 \leqq i \leqq N+2)$, we have $\tilde{f}^{j*}(H_i)=D_{k_i^*}+D_{\gamma_{ji}}(\gamma_{ji} \in K)$ for all j. This contradicts the induction hypothesis. Thus, we conclude $\# \mathscr{G} \leqq q_N$.

§4. The proof of Main Theorem.

For the proof of Main Theorem, we recall some lemmas given in the previous paper [6].

LEMMA 4.1. *Let $P(w_1,\cdots,w_{m+1})$ be a homogeneous polynomial of degree $d(\geqq 1)$ and set*

$$P(w)=\sum_{\sigma=1}^{s+1} P_\sigma(w),$$

where $P_\sigma(w)$ are nonzero monomials. Consider a meromorphic map $F=(P_1:\cdots:P_{s+1}): P^m(C) \to P^s(C)$ and assume that $\pi_i(\{P=0\})=P^{m-1}(C)$ for $1 \leqq i \leqq m+1$, where $\pi_i: P^m(C) \to P^{m-1}(C)$ is the map defined by $\pi_i((w_1:\cdots:w_{m+1}))=(w_1:\cdots:w_{i-1}:w_{i+1}:\cdots:w_{m+1})$. Then, for each point $w=(w_1:\cdots:w_{m+1})$ with $w_1 w_2 \cdots w_{m+1} \neq 0$, we have

$$\# F^{-1}(F(w)) \leqq d^m.$$

For the proof, see [6], p. 535.

LEMMA 4.2. *Let L be a holomorphic line bundle over a compact complex space X. Then, there exists a number d_L depending only on L such that, for arbitrary holomorphic sections $\phi_1,\cdots,\phi_{N+2}$ of L on X, we can find a nonzero homogeneous polynomial $P(w_1,\cdots, w_{N+2})$ of degree $\leqq d_L$ satisfying the condition*

$$P(\phi_1,\cdots,\phi_{N+2})=0.$$

For the proof, see [9].

LEMMA 4.3. *Let L be a holomorphic line bundle over an N-dimensional complex space which has at least one system of $N+1$ algebraically independent holomorphic sections. Then, there exists a constant k_L depending only on L such that for algebraically independent $\phi_1,\cdots,\phi_{N+1} \in H^0(X, \mathcal{O}(L))$ the meromorphic map $\Phi:=(\phi_1:\cdots:\phi_{N+1}): X \to P^N(C)$ satisfies the condition that $\# \Phi^{-1}(w) \leqq k_L$ for every point w in a Zariski open dense subset of $P^N(C)$.*

For the proof, see [6], p. 537.

Now, we set about the proof of Main Theorem. As in §1, we consider a holomorphic line bundle L over a compact complex space X, another complex space Y written as $Y=\bar{Y}-A$ with a compact complex space $\bar{Y}$ and an irreducible analytic subset A, and divisors $D_1,\cdots,D_{N+2} \in |\mathrm{L}|$ and $E_1,\cdots,E_{N+2}$ on Y such that $D_1,\cdots,D_{i-1},D_{i+1},\cdots,D_{N+2}$ are algebraically independent with respect to L for each $i(1 \leqq i \leqq N+2)$. Choose nonzero holomorphic sections ϕ_i with $(\phi_i)=D_i$ and define a meromorphic map

$$\Phi=(\phi_1:\cdots:\phi_{N+2}): X \to P^{N+1}(\mathrm{C}).$$

Then, $V=\Phi(X)$ is an irreducible algebraic subset of $P^{N+1}(\mathbf{C})$. By Lemma 4.2, we can find a nonzero homogeneous polynomial $P(w_1,\cdots,w_{N+2})$ of degree $\leqq d_L$ such that

$$P(\phi_1,\cdots,\phi_{N+2})=0.$$

By the assumption, $\pi_i(\{P=0\})=P^N(\mathbf{C})$ for each i $(1\leqq i\leqq N+2)$. Set $P(w)=\sum_{\sigma=1}^{s+1}P_\sigma(w)$, where each $P_\sigma(w)$ is a nonzero homogeneous monomial. We consider a meromorphic map

$$F=(P_1:\cdots:P_{s+1}):P^{N+1}(\mathbf{C})\rightarrow P^s(\mathbf{C}).$$

By virtue of Lemmas 4.2 and 4.3, $\psi:=F\circ\Phi$ has the property that $\#\psi^{-1}(\psi(w))$ is bounded by a constant depending only on L for each $w\in X$ possibly excluded a thin analytic set. Set

$$H_{s+2}=\{u_1+u_2+\cdots+u_{s+1}=0\}$$

$$H_i=\{u_i=0\}\cap H_{s+2}\quad(1\leqq i\leqq s+1)$$

in $P^s(\mathbf{C})$, where $(u_1:\cdots:u_{s+1})$ are homogeneous coordinates on $P^s(\mathbf{C})$. We identify H_{s+2} with $P^{s-1}(\mathbf{C})$. Then, $H_1,\cdots,H_{s+1}$ correspond to $s+1$ hyperplanes in $P^{s-1}(\mathbf{C})$ located in general position. To each monomial $P_\sigma(w_1,\cdots,w_{N+2})=c_\sigma w_1^{l_{\sigma 1}}\cdots w_{N+2}^{l_{\sigma N+2}}$ we correspond divisors

$$\tilde{D}_\sigma=l_{\sigma 1}D_1+\cdots+l_{\sigma N+2}D_{N+2}$$

$$\tilde{E}_\sigma=l_{\sigma 1}E_1+\cdots+l_{\sigma N+2}E_{N+2}.$$

Moreover, with each $f\in\mathscr{F}:=\mathscr{F}(E_i;D_i)$ we associate a meromophic map $\tilde{f}:=F\circ\Phi\circ f:Y\rightarrow P^{s-1}(\mathbf{C})$ and we denote the set of all $\tilde{f}$ for $f\in\mathscr{F}$ by $\tilde{\mathscr{F}}$. Then, $(F\circ\Phi)_*(H_\sigma)=\tilde{D}_\sigma$ and hence $\tilde{f}^*(H_\sigma)=\tilde{E}_\sigma(1\leqq\sigma\leqq s+1)$ for every $\tilde{f}\in\tilde{\mathscr{F}}$. On the other hand, since A is irreducible, for each $h\in H^*(X)\cap M(\bar{X})$ either h or $1/h$ has a holomorphic extension to $\bar{X}$. This leads to $H^*(X)\cap M(\bar{X})=\mathbf{C}^*$ by virtue of the maximum principle. Now, we apply Theorem 3.3 to the family $\mathscr{F}$, where we take $K=\mathbf{C}$. In the case $K=\mathbf{C}$, the notion of K-nondegeneracy is nothing but ($\mathbf{C}$-linear) nondegeneracy. Therefore, we can conclude that the number of nondegenerate meromorphic maps $\tilde{f}$ of Y into $P^{s-1}(\mathbf{C})$ such that $\tilde{f}^*(H_\sigma)=\tilde{E}_\sigma(1\leqq\sigma\leqq s+1)$ is bounded by a constant depending only on s. On the other hand, we can show by the same argument used in §5 of [6] that for a fixed $f_0\in\mathscr{F}$ $\#\{f\in\mathscr{F};F\circ\Phi\circ f=F\circ\Phi\circ f_0\}$ is bounded by a constant depending only on L. This concludes that $\#\mathscr{F}\leqq q_L$ for a constant q_L depending only on L. Thus, the proof of Main Theorem is completed.

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